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CLASS XII

12th FULLY REVISED EDITION

MATHEMATICIA

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CHAPTER 01

ALGEBRA Of MATRICES

If you have the belief that you can do it, you will acquire all the capacity to do it even if you may not have it at the beginning!

Overview

In this chapter, we shall learn

- ✓ Definition of Matrix, notation, terminologies & type of matrices
- ✓ Algebraic operations on matrices viz. addition, subtraction and multiplication
- ✓ Understanding various properties viz. commutative, associative properties for algebraic operations on matrices, Equality of matrices
- ✓ Transpose of matrix, Symmetric and skew-symmetric matrices
- ✓ Existence of inverse of a matrix, Elementary row and column operations
- ✓ Defining determinant of a square matrix (up to 3^{rd} order matrices)
- ✓ Properties of determinants
- ✓ Minors, Co-factors, Application of determinants in finding the area of Δ
- ✓ Adjoint of matrix and inverse of square matrix by determinant method
- ✓ Consistency & inconsistency of system of linear equations (two or three variable system of linear equations) and their solutions using inverse of matrix
- ✓ Real life - Application based Problems

BASIC ALGEBRA Of MATRICES

INTRODUCTION

Matrices are very powerful tools not only in the field of Maths but also in Economics, Computers, and Cryptography etc. In computer based programming, these matrices play a vital role in the projection of three-dimensional image into a two-dimensional screen, creating the realistic motion pictures. Matrices and their inverse matrices are used by a programmer for coding or encrypting a message. A message consists of a sequence of numbers in a binary format that is used for the communication. The process of coding and decoding requires coding theory that involves solving the linear equations. These equations are solved with the help of matrices. With these encryptions only, the internet is functioning and even financial institutions are able to transmit sensitive and private data securely.

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. Matrix - a basic introduction :

A matrix is an ordered rectangular array of numbers (real or complex) or functions which are known as *elements* or the *entries* of the matrix. It is denoted by the **upper case letters** *i.e.* A, B, C etc.

Consider a matrix A given as, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$.

Here in matrix A depicted above, the horizontal lines of elements are said to constitute ***rows*** of the matrix A and vertical lines of elements are said to constitute ***columns*** of the matrix.
Thus matrix A has ***m* rows** and ***n* columns**.
The array is enclosed by brackets [ ], the parentheses ( ) and the double vertical bars ‖ ‖.

- *A matrix having m rows and n columns is called a matrix of order* $m\times n$ *(read as '**m by n**' matrix). And a matrix A of order* $m\times n$ *is depicted as* $A=\left[a_{ij}\right]_{m\times n}$; $i, j \in \mathrm{N}$.
- *Also in general,* a_{ij} *means an element lying in the* i^{th} *row and* j^{th} *column.*
- *No. of elements in the matrix* $A=\left[a_{ij}\right]_{m\times n}$ *is given as* $(m)(n)$.

02. Types Of Matrices :

a) Column matrix :

A matrix having only one column is called a *column matrix* or *column vector*.

e.g. $\begin{bmatrix}0\\1\\-2\end{bmatrix}_{3\times1}, \begin{bmatrix}8\\5\end{bmatrix}_{2\times1}$.

➲ **General notation:** $\mathbf{A=[a_{ij}]_{m\times1}}$.

b) Row matrix :

A matrix having only one row is called a *row matrix* or *row vector*.

e.g. $\begin{bmatrix}-1 & 2 & \sqrt{3} & 4\end{bmatrix}_{1\times4}, \begin{bmatrix}2 & -5 & 0\end{bmatrix}_{1\times3}$

➲ **General notation:** $\mathbf{A=[a_{ij}]_{1\times n}}$.

c) Square matrix :

It is a matrix in which the number of rows is equal to the number of columns i.e., an $m\times n$ matrix is said to constitute a square matrix if $m=n$ and is known as a ***square matrix of order 'n'***.

e.g. $\begin{bmatrix}1 & 2 & 5\\3 & 7 & -4\\0 & -1 & -2\end{bmatrix}_{3\times3}$ is a square matrix of order 3.

➲ **General notation:** $\mathbf{A=[a_{ij}]_{n\times n}}$.

d) Diagonal matrix :

A square matrix $A=[a_{ij}]_{m\times m}$ is said to be a *diagonal matrix* if $a_{ij}=0$, when $i\neq j$ *i.e.,* all its non-diagonal elements are *zero.*

e.g. $\begin{bmatrix}2 & 0 & 0\\0 & 5 & 0\\0 & 0 & 4\end{bmatrix}_{3\times3}$ is a diagonal matrix of order 3.

- *Also there is **one more notation** specifically used for the **diagonal matrices**. For instance, consider the matrix depicted above, it can be also written as* $\textbf{diag}(\mathbf{2 \quad 5 \quad 4})$.
- *Note that the elements* $a_{11}, a_{22}, a_{33}, ..., a_{mm}$ *of a square matrix* $A=\left[a_{ij}\right]_{m\times n}$ *of **order m** are said to constitute the **principal diagonal** or simply **the diagonal of the square matrix A**. And these elements are known as **diagonal elements of matrix A**.*

e) Scalar matrix :

A *diagonal matrix* $A=[a_{ij}]_{m\times m}$ is said to be a *scalar matrix* if its *diagonal elements* are **equal** *i.e.,*

$$a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \text{ for some constant } k \end{cases}.$$

e.g. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}$ is a scalar matrix of order 3.

f) Unit matrix or Identity matrix :

A square matrix $A = [a_{ij}]_{m\times m}$ is said to be *identity matrix* if the element a_{ij} is given by

$$a_{ij} = \begin{cases} 1, \text{ if } & i = j \\ 0, \text{ if } & i \neq j \end{cases}.$$

A *unit matrix* can also be defined as the *scalar matrix* each of whose diagonal elements is *unity*.
We denote the identity matrix of order m by I_m or I.

e.g. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Here first identity matrix is of order 3 whereas the second is of order 2.

g) Zero matrix or Null matrix :

A matrix is said to be a *null matrix* if each of its elements is '0' (zero).
It is denoted by English alphabet 'O'.

e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

h) Horizontal matrix :

A $m\times n$ matrix is said to be a *horizontal matrix* if $m < n$ i.e., if number of rows is less than the number of columns in the matrix.

e.g. $\begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 7 \end{bmatrix}_{2\times 3}$.

i) Vertical matrix :

A $m\times n$ matrix is said to be a *vertical matrix* if $m > n$ i.e., if number of rows is more than the number of columns in the matrix.

e.g. $\begin{bmatrix} 2 & 5 \\ 0 & 7 \\ 3 & 1 \end{bmatrix}_{3\times 2}$.

j) Triangular matrices :

Lower triangular matrix

A square matrix is called a lower triangular matrix if $a_{ij} = 0$ when $i < j$.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 7 \end{bmatrix}$.

Upper triangular matrix

A square matrix is called an upper triangular matrix if $a_{ij} = 0$ when $i > j$.

e.g. $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

03. Equality of Matrices :

Two matrices A and B are said to be equal and written as A = B, if they are of the **same orders** and their **corresponding elements are identical** *i.e.* $a_{ij} = b_{ij}$ for all i and j.

That is $a_{11} = b_{11}, a_{22} = b_{22}, a_{23} = b_{23}, a_{32} = b_{32}, a_{33} = b_{33}$ etc.

04. Addition of matrices :

If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by 'A + B'.

Thus if $A = [a_{ij}], B = [b_{ij}] \quad \Rightarrow A + B = [a_{ij} + b_{ij}]$.

Properties of matrix addition :

(a) Commutative property : $A + B = B + A$

(b) Associative property : $A + (B + C) = (A + B) + C$

(c) Cancellation laws : **(i)** Left cancellation - $A + B = A + C \quad \Rightarrow B = C$

(ii) Right cancellation - $B + A = C + A \quad \Rightarrow B = C$.

05. Multiplication of a matrix by a scalar :

If an $m \times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus if $A = [a_{ij}]$ and it is multiplied by a scalar k then, $kA = [k a_{ij}]$, *i.e.,* $A = [a_{ij}] \quad \therefore kA = [k a_{ij}]$.

e.g. $A = \begin{bmatrix} 2 & -1 \\ 6 & 4 \end{bmatrix} \quad \Rightarrow 3A = 3\begin{bmatrix} 2 & -1 \\ 6 & 4 \end{bmatrix}$

$\therefore 3A = \begin{bmatrix} 6 & -3 \\ 18 & 12 \end{bmatrix}$.

WORKED OUT ILLUSTRATIVE EXAMPLES

Ex01. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{(i+2j)^2}{2}$.

Sol. Consider $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be the required matrix.

As $a_{ij} = \frac{[i+2j]^2}{2}$, so we have $a_{11} = \frac{[1+2(1)]^2}{2} = \frac{9}{2}$, $a_{12} = \frac{25}{2}$, $a_{21} = 8$, $a_{22} = 18$.

So, the required matrix is $A = \begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$.

Ex02. Find the value of a, if $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$.

Sol. We have $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

By equality of matrices, we get : a – b = –1, 2a + c = 5, 2a – b = 0 and 3c + d = 13.
Solving these equations, we get : a = 1.

Ex03. Find the matrix X, if $X + Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix}$ and $X - Y = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Sol. We have $X+Y=\begin{pmatrix}7 & 0\\ 2 & 5\end{pmatrix}$ and $X-Y=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$.

On adding these two, we get : $(X+Y)+(X-Y)=\begin{pmatrix}7 & 0\\ 2 & 5\end{pmatrix}+\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$

$\Rightarrow 2X=\begin{pmatrix}10 & 0\\ 2 & 8\end{pmatrix}$

$\therefore X=\begin{pmatrix}5 & 0\\ 1 & 4\end{pmatrix}$.

Ex04. (a) For what value (s) of x, the matrix $\begin{pmatrix}-1 & 0 & y-x\\ 0 & 0 & 0\\ 0 & x+y-6 & 5\end{pmatrix}$ is a diagonal matrix?

(b) For what value(s) of $a+x$, the matrix $\begin{pmatrix}6 & 0 & 0\\ 0 & 2a+6 & 0\\ 0 & 0 & x+3\end{pmatrix}$ is a scalar matrix?

Sol. **(a)** $\because a_{ij}=0$ if $i\neq j$ for a diagonal matrix so, $y-x=0,\ x+y-6=0$.

On solving, we get : $x=3$

(b) $\because a_{ij}=k\ \forall\ i=j$ for a scalar matrix so, $6=2a+6=x+3$.

On solving, we get : $a=0,\ x=3$.

Therefore, $a+x=0+3=3$.

Ex05. (a) If $\begin{bmatrix}2 & 5\\ 3 & -7\end{bmatrix}-A=\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix}$, then find the matrix A.

(b) If $3A-B=\begin{bmatrix}5 & 0\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 3\\ 2 & 5\end{bmatrix}$, then find the matrix A.

Sol. **(a)** Here $\begin{bmatrix}2 & 5\\ 3 & -7\end{bmatrix}-A=\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2 & 5\\ 3 & -7\end{bmatrix}-A+A=\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix}+A$

$\Rightarrow \begin{bmatrix}2 & 5\\ 3 & -7\end{bmatrix}=\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix}+A$

$\Rightarrow A=\begin{bmatrix}2 & 5\\ 3 & -7\end{bmatrix}-\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix}$

$\therefore A=\begin{bmatrix}2 & 3\\ 2 & -10\end{bmatrix}$.

(b) $3A-B=\begin{bmatrix}5 & 0\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 3\\ 2 & 5\end{bmatrix}$

$\Rightarrow 3A-B+B=\begin{bmatrix}5 & 0\\ 1 & 1\end{bmatrix}+\begin{bmatrix}4 & 3\\ 2 & 5\end{bmatrix}$

$\Rightarrow 3A=\begin{bmatrix}9 & 3\\ 3 & 6\end{bmatrix}$

$$\Rightarrow A=\frac{1}{3}\times\begin{bmatrix}9 & 3\\ 3 & 6\end{bmatrix}$$

$$\therefore A=\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}.$$

Ex06. Find a matrix A such that $2A-3B+5C=O$, where $B=\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$.

Sol. As $2A-3B+5C=O$

$$\Rightarrow 2A=3B-5C$$

$$\Rightarrow 2A=\begin{bmatrix}-6 & 6 & 0\\ 9 & 3 & 12\end{bmatrix}-\begin{bmatrix}10 & 0 & -10\\ 35 & 5 & 30\end{bmatrix}$$

$$\Rightarrow 2A=\begin{bmatrix}-16 & 6 & 10\\ -26 & -2 & -18\end{bmatrix}$$

$$\Rightarrow A=\frac{1}{2}\begin{bmatrix}-16 & 6 & 10\\ -26 & -2 & -18\end{bmatrix}$$

$$\therefore A=\begin{bmatrix}-8 & 3 & 5\\ -13 & -1 & -9\end{bmatrix}.$$

Ex07. Find the values of x and y, if $\begin{bmatrix}x^2\\ y^2\end{bmatrix}-\begin{bmatrix}5x\\ 6y\end{bmatrix}=3\begin{bmatrix}-2\\ -3\end{bmatrix}$.

Sol. We have $\begin{bmatrix}x^2\\ y^2\end{bmatrix}-\begin{bmatrix}5x\\ 6y\end{bmatrix}=3\begin{bmatrix}-2\\ -3\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}x^2-5x\\ y^2-6y\end{bmatrix}=\begin{bmatrix}-6\\ -9\end{bmatrix}$$

By equality of matrices, we get : $x^2-5x=-6$ and, $y^2-6y=-9$

$\Rightarrow x^2-5x+6=0\ldots(i)$,

$y^2-6y+9=0\ldots(ii)$

By (i), $x^2-5x+6=0$

$\Rightarrow x^2-3x-2x+6=0$

$\Rightarrow (x-2)(x-3)=0$

$\therefore x=2,3$

By (ii), $y^2-6y+9=0$

$\Rightarrow y^2-3y-3y+9=0$

$\Rightarrow (y-3)(y-3)=0$

$\therefore y=3$

Therefore, $x=2,3;\ y=3$.

Ex08. If $A=\begin{bmatrix}0 & 2\\ 3 & -4\end{bmatrix}$ and $kA=\begin{bmatrix}0 & 3a\\ 2b & 24\end{bmatrix}$, then find the values of k, a and b.

Sol. $\because kA=\begin{bmatrix}0 & 3a\\ 2b & 24\end{bmatrix}$

$$\therefore k\begin{bmatrix}0 & 2\\ 3 & -4\end{bmatrix}=\begin{bmatrix}0 & 3a\\ 2b & 24\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2k \\ 3k & -4k \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

By equality of matrices, we get : $2k = 3a,\ 3k = 2b,\ -4k = 24$

$\Rightarrow 2(-6) = 3a,\ 3(-6) = 2b,\ k = -6$

$\therefore a = -4,\ b = -9,\ k = -6$.

EXERCISE 1 A

VERY SHORT ANSWER TYPE QUESTIONS

Q01. **(a)** How many matrices of order 2×3 are possible with each entry 0 or 1?
(b) What is the number of all possible matrices of order 3×3 with each entry as 0 or 1?
(c) Write the number of all possible matrices of order 2×2 with each entry 1, 2 or 3.

Q02. If a matrix has 12 elements, what are the possible orders it can have?

Q03. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 3 \end{bmatrix}$. For what value of x, A will be a scalar matrix?

Q04. **(a)** What is the element a_{23} in the matrix $A = \lambda\left[a_{ij}\right]_{3\times3}$ s.t. $a_{ij} = \dfrac{2(9i-j)}{3}$?

(b) For a 2×2 matrix $A=[a_{ij}]$, whose elements are given by $a_{ij} = \dfrac{i}{j}$, write the value of a_{12}.

(c) Write the element a_{23} of a 3×3 matrix $A=(a_{ij})$ whose elements a_{ij} are given by $a_{ij} = \dfrac{|i-j|}{2}$.

(d) Write the element a_{12} of the matrix $A=[a_{ij}]_{2\times2}$, whose elements a_{ij} are given by $a_{ij} = e^{2ix}\sin jx$.

Q05. **(a)** Construct a matrix $[a_{ij}]_{4\times3}$ such that $a_{ij} = \dfrac{i-j}{i+j}$.

(b) Construct a 3×2 matrix B such that $b_{ij} = \dfrac{|i-2j|}{3}$.

(c) Construct a 2×3 matrix A whose elements are given by $a_{ij} = \begin{cases} i-2j, & \text{if } i>j \\ i-j, & \text{if } i=j \\ -i+3j, & \text{if } i<j \end{cases}$.

Q06. If $A = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$, then for what value of ω is A an identity matrix?

Q07. If $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$, then find $(3A-B)$.

Q08. If $A = \operatorname{diag}(1 \quad -1 \quad 2)$ and $B = \operatorname{diag}(2 \quad 3 \quad -1)$, find $3A+4B$.

Q09. Simplify: $\cos\omega\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} + \sin\omega\begin{bmatrix} \sin\omega & -\cos\omega \\ \cos\omega & \sin\omega \end{bmatrix}$.

Q10. If $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that $2A+3X=5B$.

Q11. **(a)** If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix A.

(b) If $\begin{bmatrix} -2 & 4 & -2 \\ 3 & 7 & 3 \end{bmatrix} + A = \begin{bmatrix} -1 & 2 & 6 \\ 4 & 5 & 0 \end{bmatrix}$, then find the matrix A.

Q12. Solve for the unknown variables viz. w, x, y, z, a, b, c (as the case may be) in the followings:

(a) $\begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix} = 2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$

(c) $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+a \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

(d) $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

(e) $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

Q13. **(a)** If $2\begin{pmatrix} 3 & 4 \\ 5 & x \end{pmatrix} + \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 10 & 5 \end{pmatrix}$, then find (x – y).

(b) If $A = \begin{pmatrix} 0 & 3 \\ 2 & -5 \end{pmatrix}$ and $kA = \begin{pmatrix} 0 & 4a \\ -8 & 5b \end{pmatrix}$, find the values of k and a.

(c) Find the value of (x + y) from the following matrix equation :

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}.$$

(d) Find the value of a and b, if $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$.

(e) If $\begin{pmatrix} a+4 & 3b \\ 8 & -6 \end{pmatrix} = \begin{pmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{pmatrix}$, write the value of 'a – 2b'.

EXERCISE 1 B

SHORT ANSWER TYPE QUESTIONS

Q01. **(a)** Find matrix A and B, if $2A - B = \begin{bmatrix} 4 & -6 \\ -4 & 2 \end{bmatrix}$ and $A + 2B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$.

(b) Find matrix X and Y, if $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$.

(c) Find the matrix A, if $A - B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 1 \\ 3 & 2 & -3 \end{bmatrix}$ and $A + 3B = \begin{bmatrix} 0 & 1 & 2 \\ -4 & 2 & -2 \\ -1 & 0 & -5 \end{bmatrix}$.

Q02. Solve for the unknowns x and, y in the followings :

(a) $\begin{bmatrix} x+y & 3 \\ 7 & xy \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 7 & -12 \end{bmatrix}$

(b) $\begin{bmatrix} 2x+y & 3y \\ 0 & y^2-5y \end{bmatrix} = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$.

Multiplication of two matrices

01. Def. Let $A=[a_{ij}]$ be a $m\times n$ matrix and $B=[b_{jk}]$ be a $n\times p$ matrix such that the number of columns in A is equal to the number of rows in B, then the $m\times p$ matrix $C=[c_{ik}]$ such that $C_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}$ is said to be the product of the matrices A and B in that order and it is denoted by AB *i.e.* "C = AB'.

e.g. (i) $\begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}_{2\times3} \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}_{3\times2} = \begin{pmatrix} 1\times2+(-2)\times4+3\times2 & 1\times3+(-2)\times5+3\times1 \\ (-4)\times2+2\times4+5\times2 & (-4)\times3+2\times5+5\times1 \end{pmatrix}_{2\times2} = \begin{pmatrix} 0 & -4 \\ 10 & 3 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}_{2\times2} \begin{pmatrix} 1 & -1 & 0 \\ -2 & 1 & -3 \end{pmatrix}_{2\times3} = \begin{pmatrix} 2\times1+(-1)(-2) & 2(-1)+(-1)\times1 & 2\times0+(-1)(-3) \\ 3\times1+4(-2) & 3(-1)+4\times1 & 3\times0+4(-3) \end{pmatrix}_{2\times3}$

$\Rightarrow \quad = \begin{pmatrix} 4 & -3 & 3 \\ -5 & 1 & -12 \end{pmatrix}$.

☞ For **better illustration**, we need to follow a few more **examples** (to be discussed in the class).

02. Properties of matrix multiplication :

(a) Note that the **product AB is defined *only* when** the number of columns in matrix A is equal to the number of rows in matrix B.

(b) If A and B are $m\times n$ and $n\times p$ matrices respectively then the matrix AB will be an $m\times p$ matrix *i.e.,* order of matrix AB will be $m\times p$.

(c) In the product AB, A is called the ***pre-factor*** and B is called the ***post-factor***.

(d) If two matrices A and B are such that AB is possible then it is ***not necessary*** that the product BA is also possible.

(e) If A is a $m\times n$ matrix and both AB as well as BA are defined then B will be a $n\times m$ matrix.

(f) If A is a $n\times n$ matrix and I_n be the unit matrix of 'order n' then, $A\,I_n = I_n\,A = A$.

(g) Matrix multiplication is ***associative*** *i.e.*, $A(BC)=(AB)C$.

(h) Matrix multiplication is ***distributive over the addition*** *i.e.*, $A(B+C)=AB+AC$.

✪ **Idempotent matrix :** A square matrix A is said to be an idempotent matrix if $\mathbf{A^2=A}$.

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

WORKED OUT ILLUSTRATIVE EXAMPLES

Ex01. If $A=\begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$ and $B=\begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$ and $BA=(b_{ij})$, find $b_{21}+b_{32}$.

Sol. Here $BA=\begin{pmatrix}2&3\\4&5\\2&1\end{pmatrix}\begin{pmatrix}1&-2&3\\-4&2&5\end{pmatrix}=\begin{pmatrix}2\times1+3\times(-4)&2\times(-2)+3\times2&2\times3+3\times5\\4\times1+5\times(-4)&4\times(-2)+5\times2&4\times3+5\times5\\2\times1+1\times(-4)&2\times(-2)+1\times2&2\times3+1\times5\end{pmatrix}$

$\Rightarrow BA=\begin{pmatrix}-10&2&21\\-16&2&37\\-2&-2&11\end{pmatrix}=(b_{ij})$

So, $b_{21}+b_{32}=-16+(-2)=-18$.

Ex02. Find x, if $[x\quad -5\quad -1]\begin{bmatrix}1&0&2\\0&2&1\\2&0&3\end{bmatrix}\begin{bmatrix}x\\4\\1\end{bmatrix}=O$.

Sol. We have $[x\quad -5\quad -1]\begin{bmatrix}1&0&2\\0&2&1\\2&0&3\end{bmatrix}\begin{bmatrix}x\\4\\1\end{bmatrix}=O$

$\Rightarrow \left[x\times1+(-5)\times0+(-1)\times2\quad x\times0+(-5)\times2+(-1)\times0\quad x\times2+(-5)\times1+(-1)\times3\right]\begin{bmatrix}x\\4\\1\end{bmatrix}=O$

$\Rightarrow \left[x-2\quad -10\quad 2x-8\right]\begin{bmatrix}x\\4\\1\end{bmatrix}=O$

$\Rightarrow \left[(x-2)x+(-10)\times4+(2x-8)\times1\right]=O$

$\Rightarrow \left[x^2-48\right]=[0]$

By equality of matrices, we get : $x^2-48=0$

$\therefore x=\pm4\sqrt{3}$.

Ex03. (a) Find matrix X, so that $X\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}=\begin{pmatrix}-7&-8&-9\\2&4&6\end{pmatrix}$.

(b) Find matrix A, such that $\begin{pmatrix}2&-1\\1&0\\-3&4\end{pmatrix}A=\begin{pmatrix}-1&-8\\1&-2\\9&22\end{pmatrix}$.

(c) Find matrix A, if $\begin{pmatrix}2&-1\\1&0\\-3&4\end{pmatrix}A=\begin{pmatrix}-1&-8&-10\\1&-2&-5\\9&22&15\end{pmatrix}$.

Sol. (a) Let $X=\begin{pmatrix}a&x\\c&d\end{pmatrix}$.

As $X\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}=\begin{pmatrix}-7&-8&-9\\2&4&6\end{pmatrix}$ so, $\begin{pmatrix}a&x\\c&d\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}=\begin{pmatrix}-7&-8&-9\\2&4&6\end{pmatrix}$

$\Rightarrow \begin{pmatrix}a+4x&2a+5x&3a+6x\\c+4d&2c+5d&3c+6d\end{pmatrix}=\begin{pmatrix}-7&-8&-9\\2&4&6\end{pmatrix}$

By equality of matrices, we get :

$a+4x=-7,\ 2a+5x=-8,\ 3a+6x=-9,\ c+4d=2,\ 2c+5d=4,\ 3c+6d=6$.

On solving these equations, we get : $a=1,\ x=-2,\ c=2,\ d=0$.

Hence $X=\begin{pmatrix}1 & -2\\ 2 & 0\end{pmatrix}$.

(b) Let $A=\begin{pmatrix}m & n\\ x & y\end{pmatrix}$.

Now $\begin{pmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{pmatrix}A=\begin{pmatrix}-1 & -8\\ 1 & -2\\ 9 & 22\end{pmatrix}$

$\Rightarrow \begin{pmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{pmatrix}\begin{pmatrix}m & n\\ x & y\end{pmatrix}=\begin{pmatrix}-1 & -8\\ 1 & -2\\ 9 & 22\end{pmatrix}$

$\Rightarrow \begin{pmatrix}2m-x & 2n-y\\ m & n\\ -3m+4x & -3n+4y\end{pmatrix}=\begin{pmatrix}-1 & -8\\ 1 & -2\\ 9 & 22\end{pmatrix}$

By def. of equality of matrices, we get :

$2m-x=-1,\ 2n-y=-8,\ m=1,\ n=-2,\ -3m+4x=9,\ -3n+4y=22$

So, clearly $m=1,\ n=-2,\ x=3,\ y=4$.

Hence, $A=\begin{pmatrix}1 & -2\\ 3 & 4\end{pmatrix}$.

(c) Let $A=\begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix}$. $\{\because [P]_{3\times2}[A]_{m\times n}=[Q]_{3\times3}\ \therefore m=2,\ n=3$

As $\begin{pmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{pmatrix}A=\begin{pmatrix}-1 & -8 & -10\\ 1 & -2 & -5\\ 9 & 22 & 15\end{pmatrix}$

So, $\begin{pmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix}=\begin{pmatrix}-1 & -8 & -10\\ 1 & -2 & -5\\ 9 & 22 & 15\end{pmatrix}$

$\Rightarrow \begin{pmatrix}2a-d & 2b-e & 2c-f\\ a & b & c\\ -3a+4d & -3b+4e & -3c+4f\end{pmatrix}=\begin{pmatrix}-1 & -8 & -10\\ 1 & -2 & -5\\ 9 & 22 & 15\end{pmatrix}$

By equality of matrices, we get :

$2a-d=-1,\ 2b-e=-8,\ 2c-f=-10,\ a=1,\ b=-2,\ c=-5,$

$-3a+4d=9,\ -3b+4e=22,\ -3c+4f=15$

So, $a=1,\ b=-2,\ c=-5,\ d=3,\ e=4,\ f=0$.

Therefore, $A=\begin{pmatrix}1 & -2 & -5\\ 3 & 4 & 0\end{pmatrix}$.

Ex04. If $A=\begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & -2 & 0\end{bmatrix}$ and $C=\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}$, then verify that $(A+B)C=AC+BC$.

Sol. Given $A=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}$, $B=\begin{bmatrix}0&1&1\\1&0&2\\1&-2&0\end{bmatrix}$ and $C=\begin{bmatrix}2\\-2\\3\end{bmatrix}$

Now $A+B=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}+\begin{bmatrix}0&1&1\\1&0&2\\1&-2&0\end{bmatrix}=\begin{bmatrix}0&7&8\\-5&0&10\\8&-10&0\end{bmatrix}$

$\therefore (A+B)C=\begin{bmatrix}0&7&8\\-5&0&10\\8&-10&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-14+24\\-10+0+30\\16+20+0\end{bmatrix}=\begin{bmatrix}10\\20\\36\end{bmatrix}\ldots\text{(i)}$

Also $AC=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}$

$\Rightarrow AC=\begin{bmatrix}0-12+21\\-12+0+24\\14+16+0\end{bmatrix}=\begin{bmatrix}9\\12\\30\end{bmatrix}$

and, $BC=\begin{bmatrix}0&1&1\\1&0&2\\1&-2&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}$

$\Rightarrow BC=\begin{bmatrix}0-2+3\\2+0+6\\2+4+0\end{bmatrix}=\begin{bmatrix}1\\8\\6\end{bmatrix}$

$\Rightarrow AC+BC=\begin{bmatrix}9\\12\\30\end{bmatrix}+\begin{bmatrix}1\\8\\6\end{bmatrix}=\begin{bmatrix}10\\20\\36\end{bmatrix}\ldots\text{(ii)}$

By (i) and (ii), it is clear that $(A+B)C=AC+BC$.

☞ The property, given in the above example, is the **Distributive property of matrix addition**.

EXERCISE 1 C

VERY SHORT ANSWER TYPE QUESTIONS

Q01. A matrix X has $a+b$ rows and $a+2$ columns while the matrix Y has $b+1$ rows and $a+3$ columns. Both the matrices XY and YX exist. Find the values of a and b.

Q02. If $(2\quad 1\quad 3)\begin{pmatrix}-1&0&-1\\-1&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}=A$, then write the order of matrix A.

Q03. If $A=\begin{bmatrix}1&3&2\end{bmatrix}$ and $B=\begin{bmatrix}6\\2\\3\end{bmatrix}$ then, find the matrix AB.

Q04. Give an example of two non-zero 2×2 matrices A and B such that $AB=O$.

Q05. If it is given that $A=\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ then, find A^2. Here $i=\sqrt{-1}$.

Q06. If $A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then, write the matrix A^4.

Q07. If $A=\begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}$, then find the value of A^{20}.

Q08. **(a)** Solve the matrix equation : $\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ 5 \end{bmatrix}=O$.

(b) Find the values of x, from the matrix equation : $\begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 5 \end{pmatrix}=\begin{pmatrix} 3 & 7x+y \\ 2y & 10 \end{pmatrix}$.

(c) Solve the following matrix equation for x : $\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}=O$.

(d) For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix}=O$?

(e) Solve for x : $\begin{bmatrix} x & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=O$.

Q09. Evaluate : $1-\omega^2-\kappa\eta$ if $A=\begin{bmatrix} \omega & \kappa \\ \eta & -\omega \end{bmatrix}$ satisfies the equation $A^2=I$.

Q10. **(a)** If A is a square matrix, such that $A^2=A$ then, what is the value of $(I+A)^3-7A$?

(b) If A is a square matrix, such that $A^2=I$, then find the simplified of $(A-I)^3+(A+I)^3-7A$.

Q11. For what value (s) of x, the matrix product $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$ equals an identity matrix?

EXERCISE 1 D

SHORT & LONG ANSWER TYPE QUESTIONS

Q01. If $A=\begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, $C=\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, then calculate AC, BC and (A + B) C.

Also, verify that (A + B) C = AC + BC.

☞ This property is known as the **Distributive property of matrix addition**.

Q02. Find the matrix A, if $A\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}=\begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$.

Q03. If it is known that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}A=\begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, find A.

Q04. Let $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$, $B=\begin{bmatrix}x & 1\\ y & -1\end{bmatrix}$ such that $A^2+B^2=(A+B)^2$. Find the value (s) of x and y.

Q05. If $A=\begin{bmatrix}0 & -\tan\dfrac{x}{2}\\ \tan\dfrac{x}{2} & 0\end{bmatrix}$ and I is an identity matrix then, show that

$(I+A)=(I-A)\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}$.

Q06. If $\phi(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$ then, show that $\phi(x)\phi(y)=\phi(x+y)$.

Q07. Prove that the product of matrices $\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$ and $\begin{bmatrix}\cos^2\beta & \cos\beta\sin\beta\\ \cos\beta\sin\beta & \sin^2\beta\end{bmatrix}$ is a null matrix, when θ and β differ by an odd integral multiple of $\dfrac{\pi}{2}$.

Q08. Using $1+\omega+\omega^2=0$ and $\omega^3=1$, show that : $\left(\begin{bmatrix}1 & \omega & \omega^2\\ \omega & \omega^2 & 1\\ \omega^2 & 1 & \omega\end{bmatrix}+\begin{bmatrix}\omega & \omega^2 & 1\\ \omega^2 & 1 & \omega\\ \omega & \omega^2 & 1\end{bmatrix}\right)\begin{bmatrix}1\\ \omega\\ \omega^2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

☞ The identities $\mathbf{1+\omega+\omega^2=0}$ **and** $\mathbf{\omega^3=1}$ are the **identities for complex cube root of unity**.

◘ Transpose of a matrix

01. Def. If $A=[a_{ij}]_{m\times n}$ be a matrix of order $m\times n$, then the matrix which can be obtained by interchanging the rows and columns of matrix A is said to be a ***transpose of matrix A***.
The transpose of A is denoted by A' or A^T or A^c *i.e.,* if $A=[a_{ij}]_{m\times n}$ then, $A^T=[a_{ji}]_{n\times m}$.

For example, $\begin{bmatrix}3 & 2 & 0\\ 1 & -2 & 6\end{bmatrix}^T=\begin{bmatrix}3 & 1\\ 2 & -2\\ 0 & 6\end{bmatrix}$.

➲ **Properties of Transpose of matrices :**

(a) $(A+B)^T=A^T+B^T$ **(b)** $(A-B)^T=A^T-B^T$

(c) $(A^T)^T=A$ **(d)** $(kA)^T=kA^T$ where, k is any constant

(e) $(AB)^T=B^TA^T$ **(f)** $(ABC)^T=C^TB^TA^T$

02. Symmetric matrix :
A square matrix $A=[a_{ij}]_{m\times m}$ is said to be a *symmetric matrix* if $A^T=A$.

That is, if $A=[a_{ij}]$ then, $A^T=[a_{ji}]=[a_{ij}]$ $\Rightarrow A^T=A$.

For example, $\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}$, $\begin{bmatrix}2+i & 1 & 3\\ 1 & 2 & 3+2i\\ 3 & 3+2i & 4\end{bmatrix}$.

03. Skew-symmetric matrix :

A square matrix $A=[a_{ij}]$ is said to be a *skew-symmetric matrix* if $A^T=-A$ *i.e.*, if $A=[a_{ij}]$ then,

$A^T=[a_{ji}]=-[a_{ij}] \quad \Rightarrow A^T=-A$.

For example: $\begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

04. Facts you should know :

(a) For a skew symmetric matrix, $[a_{ji}]=-[a_{ij}]$. For its diagonal elements, we have $[a_{ii}]=-[a_{ii}]$ which implies, $2[a_{ii}]=O=[0]$ (Replacing j by i).
That is, ***all the diagonal elements in a skew-symmetric matrix are zero***.

(b) The matrices AA^T and A^TA are symmetric matrices.

(c) For any square matrix A, the matrix $A+A^T$ is a symmetric matrix and $A-A^T$ is a skew-symmetric matrix *always*.

(d) Also note that ***any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix*** i.e., $A=\frac{1}{2}(P)+\frac{1}{2}(Q)$, where $P=A+A^T$ is a symmetric matrix and $Q=A-A^T$ is a skew-symmetric matrix.

05. Orthogonal matrix :

A matrix A is said to be orthogonal if $A.A^T=I$ where A^T is transpose of A.

e.g. Let $A=\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, $A^T=\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

As $AA^T=\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\times\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}=\frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so, here A is an **orthogonal matrix**.

☞ For an orthogonal matrix A, we always have **Det.(A) = ±1 i.e., $|A|=\pm 1$** (to be discussed later in the Determinants topic).

WORKED OUT ILLUSTRATIVE EXAMPLES

Ex01. (a) If A is 2×3 matrix and B is a matrix such that $A'B$ and BA' are both defined. Then what is the order of matrix B?

(b) If $A=\begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$ then, find $A'A$.

Sol. **(a)** Let order of matrix B be $m\times n$.
We know order of A' will be 3×2.
As $A'B$ is defined so, clearly $2=m$.
Also BA' is defined so, clearly $n=3$.
Therefore, order of matrix B is 2×3.

(b) We have $A=\begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$

$\Rightarrow A' = \begin{bmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{bmatrix}$.

$\therefore A'A = \begin{bmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{bmatrix}\begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$

$\Rightarrow A'A = \begin{bmatrix} \sin^2 x + \cos^2 x & \sin x \cos x - \cos x \sin x \\ \cos x \sin x - \sin x \cos x & \cos^2 x + \sin^2 x \end{bmatrix}$

Therefore, $A'A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

Ex02. If $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, find α satisfying $0 < \alpha < \frac{\pi}{2}$ when $A + A^T = \sqrt{2}\, I_2$; where A^T is the transpose of A.

Sol. Here $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

$\Rightarrow A^T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

Since $A + A^T = \sqrt{2}\, I_2$

$\therefore \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} + \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \sqrt{2}\, I$

$\Rightarrow \begin{pmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$

By def. of equality of matrices, we get : $2\cos\alpha = \sqrt{2}$

$\Rightarrow \cos\alpha = \frac{1}{\sqrt{2}}$

$\therefore \alpha = \frac{\pi}{4}$.

Ex03. Show that all the diagonal elements of a skew symmetric matrix are zero.

Sol. Let $A = [a_{ij}]$ be a square matrix such that it is skew-symmetric.

So, $A = -A^T$.

That is, $[a_{ij}] = -[a_{ji}]$.

For its diagonal elements, we have $[a_{ii}] = -[a_{ii}]$ which implies, $2[a_{ii}] = O$ (Replacing j by i).

$\Rightarrow [a_{ii}] = [0]$

$\therefore a_{ii} = 0$.

Hence, all the diagonal elements of a skew symmetric matrix are zero.

Ex04. Express $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Sol. We've $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$

$$\Rightarrow A^T = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\therefore A + A^T = \begin{bmatrix} 6 & 6 & 5 \\ 6 & 10 & 7 \\ 5 & 7 & 10 \end{bmatrix} \text{ and } A - A^T = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Let $P = \frac{1}{2}\left(A + A^T\right) = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix}$ and $Q = \frac{1}{2}\left(A - A^T\right) = \begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}$.

We observe that, $P^T = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix} = P$

$\therefore$ P is symmetric matrix.

and, $Q^T = \begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -1/2 \\ -1 & 0 & 1/2 \\ 1/2 & -1/2 & 0 \end{bmatrix}$

$$\Rightarrow Q^T = -\begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix} = -Q$$

$\therefore$ Q is skew-symmetric matrix.

Hence we have, $P + Q = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} = A$.

Thus, we've expressed matrix A as the sum of a symmetric matrix and a skew-symmetric matrix.

Ex05. If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute. (A and B commute means $AB = BA$).

Sol. Given that A and B are both symmetric matrices

$\therefore A = A^T$ and $B = B^T \ldots$(i)

Let $P = AB$

$\Rightarrow P^T = (AB)^T$

$\Rightarrow P^T = B^T A^T$

$\Rightarrow P^T = BA$ (By (i))

If A and B commute then, $AB = BA$

$\therefore P^T = AB$

$\Rightarrow P^T = P$

So, P is symmetric matrix.

Ex06. If the matrix $A = \begin{bmatrix} \mathbf{0} & \mathbf{a} & \mathbf{-3} \\ \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{b} & \mathbf{1} & \mathbf{0} \end{bmatrix}$ is skew symmetric, find the value of 'a' and 'b'.

Sol. As A is skew symmetric so, $A = -A^T$ i.e., $a_{ij} = -a_{ji}$, if $A = \left[a_{ij}\right]$.

Therefore, $a_{12} = -a_{21} \Rightarrow a = -2$

and, $a_{31} = -a_{13} \Rightarrow b = -(-3) = 3$.

Hence value of a is –2 and value of b is 3.

Ex07. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{pmatrix}$ is matrix satisfying $AA' = 9I$, find x.

Sol. As $AA' = 9I$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & x & -1 \end{pmatrix} = 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 9 & 4+2x & 0 \\ 4+2x & 5+x^2 & -2-x \\ 0 & -2-x & 9 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

By equality of matrices, we get : $4+2x = 0,\ 5+x^2 = 9,\ -2-x = 0$

On solving these, we've : $x = -2$ (which satisfies the given condition).

EXERCISE 1 E

VERY SHORT ANSWER TYPE QUESTIONS

Q01. If $A = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$ and $A + A^T = I_2$ then, what is the value of x?

Q02. If $A = \begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$ then, verify that $A'A = I$.

Q03. **(a)** If A is a matrix of 2×3 and B is of 3×5, what is the order of $(AB)^T$?

(b) If A is 3×4 matrix and B is a matrix such that A^TB and BA^T are both defined. Then what is the order of matrix B?

Q04. **(a)** Write the values of p and q such that the matrix A is skew symmetric, $A = \begin{pmatrix} 0 & 5 & -3 \\ -5 & p & 4 \\ q & -4 & 0 \end{pmatrix}$.

(b) Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is given to be symmetric, find the values of a and b.

(c) If the matrix $A = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$ is written as $A = P + Q$, where P is a symmetric matrix and Q is skew symmetric matrix, then write the matrix P.

Q05. Prove that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Q06. If A and B are symmetric matrices, prove that $AB - BA$ is a skew-symmetric matrix.

Q07. Show that the matrix B^TAB is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

Q08. If B is skew-symmetric matrix, write whether ABA' is symmetric or skew-symmetric.

Q09. Show that the elements on the main diagonal of a skew-symmetric matrix are all zero.

EXERCISE 1 F

SHORT & LONG ANSWER TYPE QUESTIONS

Q01. If $A=\begin{bmatrix}2 & -1 & 2\\ -4 & 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\ 1 & 3\\ 4 & 5\end{bmatrix}$ then, verify that $(AB)^T=B^TA^T$.

Q02. If l_i, m_i, n_i; $i=1, 2, 3$ denote the direction cosines of three mutually perpendicular lines in the space, then prove that $AA^T=I$ such that $A=\begin{bmatrix}l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{bmatrix}$.

(Based on the concept from **Three Dimensional Geometry**, NCERT Chapter 11.)

Q03. If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}$ is a matrix satisfying $AA^T=9I_3$, then find the values of a and b.

Q04. Define a symmetric and skew-symmetric matrix.

Prove that for the matrix X, $X-X^T$ is skew-symmetric matrix whereas $X+X^T$, XX^T and X^TX is symmetric matrix, where $X=\begin{bmatrix}-1 & 1\\ 2 & -4\end{bmatrix}$.

Q05. If $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}$, show that AB – BA is a skew-symmetric matrix.

Q06. Find $\frac{1}{2}(A+A')$ and $\frac{1}{2}(A-A')$, where $A=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$.

Q07. Express the matrix $\begin{bmatrix}2 & -1\\ 4 & 5\end{bmatrix}$ as the sum of symmetric and skew-symmetric matrix.

Q08. **(a)** Express $\begin{bmatrix}3 & 2 & 3\\ 4 & 5 & 3\\ 2 & 4 & 5\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

(b) Express $A=\begin{bmatrix}-3 & 6 & 0\\ 4 & -5 & 8\\ 0 & -7 & -1\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

(c) Express the matrix $\begin{bmatrix}2 & 4 & -6\\ 7 & 3 & 5\\ 1 & -2 & 4\end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrix.

Q09. Let A and B be a 3×3 matrix. Let $A+B=2B^T$ and $3A+2B=I_3$.

Determine the value of the matrix sum $10A+5B$.

Invertible Matrix

01. Def. If A is a square matrix of order *m* and if there exists another square matrix B of the same order *m*, such that AB = BA = I, then B is called the inverse matrix of A and it is denoted by A^{-1}. A matrix having an inverse is said to be ***invertible***.

It is to note that if B is inverse of A, then A is also the inverse of B. In other words, if it is known that $AB = BA = I$ *then,* $A^{-1} = B \Leftrightarrow B^{-1} = A$.

02. Elementary Operations or Transformations of a Matrix :

The following three operations applied on the row (or column) of a matrix are called elementary row (or column) transformations-

(a) Interchange of any two rows (or columns) **:** When i^{th} row (or column) of a matrix is interchanged with the j^{th} row (or column), it is denoted as $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$).

(b) Multiplying all elements of a row (or column) of a matrix by a non-zero scalar **:** When the i^{th} row (or column) of a matrix is multiplied by a scalar k, it is denoted as $R_i \to kR_i$ (or $C_i \to kC_i$).

(c) Adding to the elements of a row (or column), the corresponding elements of any other row (or column) multiplied by any scalar k**:** When k times the elements of j^{th} row (or column) is added to the corresponding elements of the i^{th} row (or column), it is denoted as $R_i \to R_i + kR_j$ (or $C_i \to C_i + kC_j$).

<u>NOTE</u>: In case, after applying one or more elementary row (or column) operations on A = IA (or A = AI), if we obtain **all zeros in one or more rows** of the matrix A on LHS, then $\mathbf{A^{-1}}$ **does not exist.**

03. Inverse or Reciprocal of a square matrix :

If A is a square matrix of order n, then a matrix B (if such a matrix exists) is called the inverse of A if $AB = BA = I_n$.

Also note that the inverse of a square matrix A is denoted by A^{-1} and we write, $A^{-1} = B$.

- *Inverse of a square matrix A exists if and only if A is non-singular matrix i.e.,* $|A| \neq 0$ (*explained later in the Determinant section*).
- *If B is inverse of A, then A is also the inverse of B.*

04. Algorithm to find Inverse of a matrix by Elementary Operations or Transformations :

➲ **By Row Transformations :**

STEP 1 - Write the given square matrix as $A = I_nA$.

STEP 2 - Perform a sequence of elementary row operations successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result $I_n = BA$.

STEP 3 - Matrix B is the inverse of A. So, write $A^{-1} = B$.

➲ **By Column Transformations :**

STEP 1 - Write the given square matrix as $A = AI_n$.

STEP 2 - Perform a sequence of elementary column operations successively on A on the LHS and post-factor I_n on the RHS till we obtain the result $I_n = AB$.

STEP 3 - Matrix B is the inverse of A. So, write $A^{-1} = B$.

WORKED OUT ILLUSTRATIVE EXAMPLES

Ex01. In the following matrix equation use elementary operation $R_2 \to R_2 + R_1$ and, write the equation thus obtained : $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -3 \\ 9 & -4 \end{pmatrix}$.

Sol. $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -3 \\ 9 & -4 \end{pmatrix}$

Applying $R_2 \to R_2 + R_1$, we get : $\begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -3 \\ 17 & -7 \end{pmatrix}$.

$\because$ in row operations, we have $A = IA$.

Ex02. Using elementary operations, find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.

Sol. Here $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

Using elementary row operations, $A = IA$ i.e., $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A$

Applying $R_2 \to R_2 - 2R_1$, $\begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} A$

Applying $R_2 \to -\frac{1}{5} R_2$, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2/5 & -1/5 \end{pmatrix} A$

Applying $R_1 \to R_1 - 2R_2$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix} A$

$\therefore A^{-1} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}$ i.e., $\frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ $(\because I = A^{-1}A$.

Ex03. Find the inverse (A^{-1}) of the matrix $A = \begin{pmatrix} -1 & 4 \\ 7 & 20 \end{pmatrix}$, using elementary operations.

Sol. Using elementary column operations, we have : $A = AI$

$\therefore \begin{pmatrix} -1 & 4 \\ 7 & 20 \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

By $C_1 \to (-1)C_1$, $\begin{pmatrix} 1 & 4 \\ -7 & 20 \end{pmatrix} = A\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

By $C_2 \to C_2 - 4C_1$, $\begin{pmatrix} 1 & 0 \\ -7 & 48 \end{pmatrix} = A\begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}$

By $C_2 \to \frac{1}{48} C_2$, $\begin{pmatrix} 1 & 0 \\ -7 & 1 \end{pmatrix} = A\begin{pmatrix} -1 & \frac{1}{12} \\ 0 & \frac{1}{48} \end{pmatrix}$

By $C_1 \to C_1 + 7C_2$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A\begin{pmatrix} -\frac{5}{12} & \frac{1}{12} \\ \frac{7}{48} & \frac{1}{48} \end{pmatrix}$

As $I = AA^{-1}$ $\therefore A^{-1} = \begin{pmatrix} -\frac{5}{12} & \frac{1}{12} \\ \frac{7}{48} & \frac{1}{48} \end{pmatrix} = \frac{1}{48}\begin{pmatrix} -20 & 4 \\ 7 & 1 \end{pmatrix}.$

Ex04. Using elementary transformations, find the inverse of the matrix : $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$.

Sol. Let $A = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$

By using elementary row transformations, we have : $A = IA$

$$\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A \qquad \text{(Applying } R_2 \to R_2 + 3R_1, R_3 \to R_3 - 2R_1$$

$$\begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -7 \\ 0 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}A \qquad \text{(Applying } R_2 \to \frac{1}{9}R_2$$

$$\begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -7/9 \\ 0 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1/9 & 0 \\ -2 & 0 & 1 \end{pmatrix}A \qquad \text{(Applying } R_1 \to R_1 - 3R_2, R_3 \to R_3 + 5R_2$$

$$\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -7/9 \\ 0 & 0 & 1/9 \end{pmatrix} = \begin{pmatrix} 0 & -1/3 & 0 \\ 1/3 & 1/9 & 0 \\ -1/3 & 5/9 & 1 \end{pmatrix}A \qquad \text{(Applying } R_3 \to 9R_3$$

$$\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -7/9 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1/3 & 0 \\ 1/3 & 1/9 & 0 \\ -3 & 5 & 9 \end{pmatrix}A \qquad \text{(Applying } R_1 \to R_1 - \frac{1}{3}R_3, R_2 \to R_2 + \frac{7}{9}R_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix}A$$

$$\Rightarrow I = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix}A$$

So, $A^{-1} = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix}$. $(\because I = A^{-1}A$

Hence, the inverse of matrix $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix}$.

Ex05. Using elementary transformations, find the inverse of the matrix : $\begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-2} \\ \mathbf{-3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} \end{pmatrix}$**.**

Sol. By using elementary row operations, we have $A = IA$

$$\therefore \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

By $R_1 \leftrightarrow R_3$

$$\Rightarrow \begin{bmatrix} 3 & 7 & 2 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

By $R_1 \to R_1 - R_3,\ R_2 \to R_2 - R_3$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

By $R_3 \to R_3 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \\ 0 & -5 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix} A$$

By $R_3 \to R_3 + 5R_2$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -2 & 5 & -2 \end{bmatrix} A$$

By $R_1 \to R_1 - 4R_2,\ R_3 \to -R_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix} A$$

By $R_1 \to R_1 - R_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix}. \qquad (\text{As } I = A^{-1}A$$

Ex06. Using elementary row transformations, find the inverse of the matrix
$\begin{bmatrix} \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{2} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{4} & \mathbf{1} \end{bmatrix}$**.**

Sol. Let $A=\begin{bmatrix}3&0&-1\\2&3&0\\0&4&1\end{bmatrix}$.

By row transformations, $A=IA$

$$\begin{bmatrix}3&0&-1\\2&3&0\\0&4&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A$$

By $R_1\to R_1-R_2$

$$\Rightarrow\begin{bmatrix}1&-3&-1\\2&3&0\\0&4&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}A$$

By $R_2\to R_2-2R_1$

$$\Rightarrow\begin{bmatrix}1&-3&-1\\0&9&2\\0&4&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\-2&3&0\\0&0&1\end{bmatrix}A$$

By $R_2\to R_2-2R_3$

$$\Rightarrow\begin{bmatrix}1&-3&-1\\0&1&0\\0&4&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\-2&3&-2\\0&0&1\end{bmatrix}A$$

By $R_3\to R_3-4R_2$

$$\Rightarrow\begin{bmatrix}1&-3&-1\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\-2&3&-2\\8&-12&9\end{bmatrix}A$$

By $R_1\to R_1+R_3$

$$\Rightarrow\begin{bmatrix}1&-3&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}9&-13&9\\-2&3&-2\\8&-12&9\end{bmatrix}A$$

By $R_1\to R_1+3R_2$

$$\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&-4&3\\-2&3&-2\\8&-12&9\end{bmatrix}A \qquad (\because I=A^{-1}A$$

Hence, $A^{-1}=\begin{bmatrix}3&-4&3\\-2&3&-2\\8&-12&9\end{bmatrix}$.

Ex07. If $A=\begin{bmatrix}2&1&1\\1&0&1\\0&2&-1\end{bmatrix}$, find the inverse of A using elementary row transformations and hence solve the matrix equation : $XA=\begin{bmatrix}1&0&1\end{bmatrix}$.

Sol. Here $A=\begin{bmatrix}2&1&1\\1&0&1\\0&2&-1\end{bmatrix}$

By using row transformations, we know $A = IA$

So, $\begin{bmatrix}2&1&1\\1&0&1\\0&2&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A$

By $R_1 \leftrightarrow R_2$, $\begin{bmatrix}1&0&1\\2&1&1\\0&2&-1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}A$

By $R_2 \to R_2-2R_1$, $\begin{bmatrix}1&0&1\\0&1&-1\\0&2&-1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&-2&0\\0&0&1\end{bmatrix}A$

By $R_3 \to R_3-2R_2$, $\begin{bmatrix}1&0&1\\0&1&-1\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&-2&0\\-2&4&1\end{bmatrix}A$

By $R_2 \to R_2+R_3$, $R_1 \to R_1-R_3$, $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2&-3&-1\\-1&2&1\\-2&4&1\end{bmatrix}A$

Since $I = A^{-1}A$

$\therefore A^{-1}=\begin{bmatrix}2&-3&-1\\-1&2&1\\-2&4&1\end{bmatrix}$.

Now $XA=\begin{bmatrix}1&0&1\end{bmatrix}$

$\Rightarrow XAA^{-1}=\begin{bmatrix}1&0&1\end{bmatrix}A^{-1}$

$\Rightarrow XI=\begin{bmatrix}1&0&1\end{bmatrix}A^{-1}$

$\Rightarrow X=\begin{bmatrix}1&0&1\end{bmatrix}\begin{bmatrix}2&-3&-1\\-1&2&1\\-2&4&1\end{bmatrix}$

$\therefore X=\begin{bmatrix}0&1&0\end{bmatrix}$.

Ex08. (a) If $A=\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}$, then find A^2-5A.

(b) If $A=\begin{bmatrix}4&2\\-1&1\end{bmatrix}$, show that $(A-2I)(A-3I)=O$.

Sol. **(a)** $A^2=A.A=\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}$

$$\therefore A^2 - 5A = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$$

Therefore, $A^2 - 5A = \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix}$.

(b) LHS : $(A-2I)(A-3I) = \left\{\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\left\{\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O = \text{RHS}.$$

Ex09. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and I is the identity matrix of order 2, then show that $A^2 = 4A - 3I$. Hence, find A^{-1}.

Sol. We have $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\Rightarrow A^2 = A.A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \ldots\text{(i)}$$

Also $4A - 3I = 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \ldots\text{(ii)}$

By (i) and (ii), we get : $A^2 = 4A - 3I$.

Pre-multiplying both sides by A^{-1} we get : $A^{-1}AA = 4A^{-1}A - 3A^{-1}I$

$\Rightarrow IA = 4I - 3A^{-1}$

$$\Rightarrow 3A^{-1} = 4I - A = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Ex10. If $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and $A^2 - \lambda A + \mu I = O$, then find the values of λ and μ.

Sol. We have $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$$\therefore A^2 = AA = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2\times2+(-1)\times(-1) & 2\times(-1)+(-1)\times2 \\ (-1)\times2+2\times(-1) & (-1)\times(-1)+2\times2 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

Now we also have $A^2 - \lambda A + \mu I = O$

$$\therefore \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} - \lambda\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + \mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} - \begin{pmatrix} 2\lambda & -\lambda \\ -\lambda & 2\lambda \end{pmatrix} + \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 5-2\lambda+\mu & -4+\lambda \\ -4+\lambda & 5-2\lambda+\mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

By equality of matrices, we get : $5-2\lambda+\mu=0,\ -4+\lambda=0$

$\Rightarrow \mu = 2\lambda - 5,\ \lambda = 4$

Hence, $\lambda = 4$ and $\mu = 3$.

EXERCISE 1 G

VERY SHORT ANSWER TYPE QUESTIONS

Q01. Find the matrix A, if it is given that $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} A = A^2$.

Q02. Use elementary column operations $C_2 \to C_2 - 2C_1$ in the matrix equation $\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

Q03. **(a)** If $Z = \begin{Vmatrix} 10 & -2 \\ -5 & 1 \end{Vmatrix}$ then find Z^{-1}, if it exists. Use elementary operations.

(b) If $A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$ then find inverse of A, if it exists. Use elementary row operations.

EXERCISE 1 H

SHORT & LONG ANSWER TYPE QUESTIONS

Q01. Find the value of x, y and z, if $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A' = A^{-1}$.

OR Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies the equation $A^T A = I$.

Q02. Show that $\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $x^2 - 3x - 7 = 0$.

Thus find the inverse of given matrix.

Q03. **(a)** If $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ then, prove that $(A-5I)(A-2I) = O$. Hence find A^{-1}.

(b) For the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$, show that $A^2 - 5A + 4I = O$. Hence find A^{-1}.

(c) If $A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$, then show that $A^2-4A-5I=O$, and hence find A^{-1}.

Q04. For the matrix $A=\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}$, verify that $A^3-6A^2+5A+11\,I=O$. Hence find A^{-1}.

Q05. If $A=\begin{bmatrix}3&1\\7&5\end{bmatrix}$ then, find x and y so that $A^2+xI-yA=O$. Hence find A^{-1}.

Q06. If $A=\begin{bmatrix}3&2\\1&1\end{bmatrix}$ then, find a and b so that $A^2+aA+bI=O$. Hence find A^{-1}.

Q07. By using **elementary operations (transformations)**, find the inverse of matrix A (if it exists) in the followings :

(a) $\begin{bmatrix}1&2\\2&-1\end{bmatrix}$ **(b)** $\begin{bmatrix}3&10\\2&7\end{bmatrix}$ **(c)** $\begin{pmatrix}1&-1\\2&3\end{pmatrix}$

(d) $\begin{bmatrix}6&-3\\-2&1\end{bmatrix}$ **(e)** $\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$ **(f)** $\begin{bmatrix}1&-1&2\\0&2&-3\\3&-2&4\end{bmatrix}$

(g) $\begin{bmatrix}2&-3&3\\2&2&3\\3&-2&2\end{bmatrix}$ **(h)** $\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$ **(i)** $\begin{bmatrix}0&0&-1\\3&4&5\\-2&-4&-7\end{bmatrix}$

(j) $\begin{pmatrix}1&3&-2\\-3&0&-1\\2&1&0\end{pmatrix}$ **(k)** $\begin{pmatrix}1&-1&0\\2&5&3\\0&2&1\end{pmatrix}$ **(l)** $\begin{pmatrix}2&0&-1\\5&1&0\\0&1&3\end{pmatrix}$

MISCELLANEOUS EXAMPLES

Ex01. If $A=\begin{bmatrix}3&1\\-1&2\end{bmatrix}$ then, show that $A^2-5A+7I=O$.

Sol. We have $A=\begin{bmatrix}3&1\\-1&2\end{bmatrix}$

$\therefore A^2=A.A=\begin{bmatrix}3&1\\-1&2\end{bmatrix}\begin{bmatrix}3&1\\-1&2\end{bmatrix}$

$\Rightarrow A^2=\begin{bmatrix}9-1&3+2\\-3-2&-1+4\end{bmatrix}=\begin{bmatrix}8&5\\-5&3\end{bmatrix}$ ...(i)

$-5A=-5\begin{bmatrix}3&1\\-1&2\end{bmatrix}=\begin{bmatrix}-15&-5\\5&-10\end{bmatrix}$ ...(ii)

And, $7I=7\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}7&0\\0&7\end{bmatrix}$ ...(iii)

Adding these three equations, we get : $A^2-5A+7I=\begin{bmatrix}8&5\\-5&3\end{bmatrix}+\begin{bmatrix}-15&-5\\5&-10\end{bmatrix}+\begin{bmatrix}7&0\\0&7\end{bmatrix}$

$$\Rightarrow A^2-5A+7I=\begin{bmatrix}8-15+7 & 5-5+0\\ -5+5+0 & 3-10+7\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$\therefore A^2-5A+7I=O$. **H. P.**

Ex02. Show that for the matrix $A=\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}$, $A^3-6A^2+5A+11\,I=O$. Hence, find A^{-1}.

Sol. We have $A^2=\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}=\begin{bmatrix}4&2&1\\-3&8&-14\\7&-3&14\end{bmatrix}$,

$$A^3=\begin{bmatrix}4&2&1\\-3&8&-14\\7&-3&14\end{bmatrix}\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}=\begin{bmatrix}8&7&1\\-23&27&-69\\32&-13&58\end{bmatrix}.$$

LHS : $A^3-6A^2+5A+11\,I$

$$\Rightarrow \quad =\begin{bmatrix}8&7&1\\-23&27&-69\\32&-13&58\end{bmatrix}-6\begin{bmatrix}4&2&1\\-3&8&-14\\7&-3&14\end{bmatrix}+5\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}+11\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\Rightarrow \quad =\begin{bmatrix}8&7&1\\-23&27&-69\\32&-13&58\end{bmatrix}-\begin{bmatrix}24&12&6\\-18&48&-84\\42&-18&84\end{bmatrix}+\begin{bmatrix}5&5&5\\5&10&-15\\10&-5&15\end{bmatrix}+\begin{bmatrix}11&0&0\\0&11&0\\0&0&11\end{bmatrix}$$

$$\Rightarrow \quad =\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O=\text{RHS}.$$

Now $A^3-6A^2+5A+11\,I=O$

Pre-multiplying by A^{-1} both sides, we get : $A^{-1}AA^2-6A^{-1}AA+5A^{-1}A+11\,A^{-1}I=A^{-1}O$

$\Rightarrow IA^2-6\,IA+5\,I+11\,A^{-1}=O$

$$\Rightarrow A^{-1}=\frac{6A-5I-A^2}{11}$$

$$\Rightarrow A^{-1}=\frac{1}{11}\left\{\begin{bmatrix}6&6&6\\6&12&-18\\12&-6&18\end{bmatrix}-\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}-\begin{bmatrix}4&2&1\\-3&8&-14\\7&-3&14\end{bmatrix}\right\}$$

$$\therefore A^{-1}=\frac{1}{11}\begin{bmatrix}-3&4&5\\9&-1&-4\\5&-3&-1\end{bmatrix}.$$

Ex03. If $A=\begin{pmatrix}1&0&2\\0&2&1\\2&0&3\end{pmatrix}$ and $A^3-6A^2+7A+k\,I_3=O$, find k.

Sol. We have $A=\begin{pmatrix}1&0&2\\0&2&1\\2&0&3\end{pmatrix}$

$$\Rightarrow A^2 = AA = \begin{pmatrix}1&0&2\\0&2&1\\2&0&3\end{pmatrix}\begin{pmatrix}1&0&2\\0&2&1\\2&0&3\end{pmatrix} = \begin{pmatrix}5&0&8\\2&4&5\\8&0&13\end{pmatrix}$$

$$\text{Also, } A^3 = AA^2 = \begin{pmatrix}1&0&2\\0&2&1\\2&0&3\end{pmatrix}\begin{pmatrix}5&0&8\\2&4&5\\8&0&13\end{pmatrix} = \begin{pmatrix}21&0&34\\12&8&23\\34&0&55\end{pmatrix}$$

Now $A^3 - 6A^2 + 7A + kI_3 = O$

$$\Rightarrow \begin{pmatrix}21&0&34\\12&8&23\\34&0&55\end{pmatrix} - 6\begin{pmatrix}5&0&8\\2&4&5\\8&0&13\end{pmatrix} + 7\begin{pmatrix}1&0&2\\0&2&1\\2&0&3\end{pmatrix} + k\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = O$$

$$\Rightarrow \begin{pmatrix}21&0&34\\12&8&23\\34&0&55\end{pmatrix} - \begin{pmatrix}30&0&48\\12&24&30\\48&0&78\end{pmatrix} + \begin{pmatrix}7&0&14\\0&14&7\\14&0&21\end{pmatrix} + \begin{pmatrix}k&0&0\\0&k&0\\0&0&k\end{pmatrix} = O$$

$$\Rightarrow \begin{pmatrix}-2&0&0\\0&-2&0\\0&0&-2\end{pmatrix} + \begin{pmatrix}k&0&0\\0&k&0\\0&0&k\end{pmatrix} = O$$

$$\Rightarrow \begin{pmatrix}k-2&0&0\\0&k-2&0\\0&0&k-2\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

By equality of matrices, we get : $k-2=0$ $\quad\therefore k=2$.

Ex04. If $A = \begin{bmatrix}3&-4\\1&-1\end{bmatrix}$, then show that $A^n = \begin{bmatrix}1+2n&-4n\\n&1-2n\end{bmatrix}$ $\forall\, n \in N$.

Sol. We shall be using principle of mathematical induction to prove this.

Let P(n) : $A^n = \begin{bmatrix}1+2n&-4n\\n&1-2n\end{bmatrix} \forall n \in N$

For n = 1, P(1) : $A^1 = \begin{bmatrix}1+2(1)&-4(1)\\1&1-2(1)\end{bmatrix} = \begin{bmatrix}3&-4\\1&-1\end{bmatrix} = A$ $\qquad$ (Given $A = \begin{bmatrix}3&-4\\1&-1\end{bmatrix}$

$\therefore$ P(1) is true.

Assume that P(k) is true for $k \in N$ i.e., P(k) : $A^k = \begin{bmatrix}1+2k&-4k\\k&1-2k\end{bmatrix}$ $\forall k \in N \ldots$(i)

We have to show that P(k + 1) is also true whenever P(k) is true i.e.,

P(k + 1) : $A^{k+1} = \begin{bmatrix}1+2(k+1)&-4(k+1)\\k+1&1-2(k+1)\end{bmatrix} = \begin{bmatrix}3+2k&-4-4k\\k+1&-1-2k\end{bmatrix}$

Consider LHS : $A^{k+1} = A^kA = \begin{bmatrix}1+2k&-4k\\k&1-2k\end{bmatrix}\begin{bmatrix}3&-4\\1&-1\end{bmatrix}$ $\qquad$ (By (i))

$$= \begin{bmatrix}3+6k-4k&-4-8k+4k\\3k+1-2k&-4k-1+2k\end{bmatrix}$$

$$= \begin{bmatrix}3+2k&-4-4k\\k+1&-1-2k\end{bmatrix} = \text{RHS.}$$

$\therefore$ P(k + 1) is also true.

Hence by principle of mathematical induction, P(n) is always true for all natural numbers n.

Ex05. Let $A=\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}$, then show that $A^2-4A+7I=O$. Using this result, calculate A^3 also.

Sol. Here $A=\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}$

$$\Rightarrow A^2=A.A=\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}1 & 12\\ -4 & 1\end{pmatrix}$$

$$\Rightarrow A^2-4A+7I=\begin{pmatrix}1 & 12\\ -4 & 1\end{pmatrix}-4\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}+7\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\therefore A^2-4A+7I=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=O.$$

Now $A^2-4A+7I=O$

$\Rightarrow A^2=4A-7I$

$\Rightarrow A.A^2=A(4A-7I)$ (Pre-multiplying by A both sides

$\Rightarrow A^3=4A^2-7AI=4(4A-7I)-7A$ (Using $A^2=4A-7I$

$\therefore A^3=16A-28I-7A=9A-28I$

Therefore, $A^3=9A-28I=\begin{pmatrix}18 & 27\\ -9 & 18\end{pmatrix}-\begin{pmatrix}28 & 0\\ 0 & 28\end{pmatrix}=\begin{pmatrix}-10 & 27\\ -9 & -10\end{pmatrix}$.

Ex06. If $A=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$, then find A^2+3I. Hence, find A^4.

Sol. As $A^2=A.A=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}$

Therefore, $A^2+3I=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}+3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$$\Rightarrow A^2+3I=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}+\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=O.$$

Now $A^2+3I=O$

$\Rightarrow A^2=-3I$

$\because A^4=A^2.A^2$

$\therefore A^4=(-3I).(-3I)=9I.I=9I=\begin{pmatrix}9 & 0\\ 0 & 9\end{pmatrix}$.

Ex07. If $A=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$, find A^2 and show that $A^2=A^{-1}$.

Sol. $A^2=AA=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$

$$\Rightarrow A^2=\begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{bmatrix}$$

As $A^2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Now $A^2A = I$

Post-multiplying both sides by A^{-1}, we get : $A^2AA^{-1} = IA^{-1}$

$\Rightarrow A^2I = A^{-1}$

$\therefore A^2 = A^{-1}$.

EXERCISE 1 I

SHORT & LONG ANSWER TYPE QUESTIONS

Q01. **(a)** If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find the value of k so that $A^2 = kA - 2I$.

(b) If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $A^{-1} = kA$, find the value of k.

(c) If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19}A$.

Q02. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ and $f(x) = x^2 - 5x - 6$ then, find $f(A)$.

Q03. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ and $f(x) = x^2 - 5x + 6$ then, find $f(A)$.

OR If $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$, find $A^2 - 5A + 16I$. (This sum is **similar** to the question given above).

Q04. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ then, show that A is a root of the cubic equation $x^3 - 6x^2 + 7x + 2 = 0$.

Q05. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ then, prove that $A^3 - 23A - 40I = O$.

Q06. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$. Show that $f(A) = O$. Use this result to find A^5.

Q07. If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and $A^2 - 2B + 7I = O$ then, find the matrix B.

Q08. If $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X s. t. $A^2 - 5A + 4I + X = O$.

Q09. Prove the followings by the **Principle Of Mathematical Induction** :

(a) $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in N$ if $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

(b) $A^n = \begin{bmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{bmatrix}$ for all $n \in N$, if $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$.

(c) $(aI + bA)^n = a^n I + n a^{n-1} b A \ \forall\ n \in N$, where I is the identity matrix of 2^{nd} order, if it is given that matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(d) $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$, $n \in N$ if $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(e) $A^n = \begin{bmatrix} a^n & \dfrac{b(a^n - 1)}{a-1} \\ 0 & 1 \end{bmatrix}$, $n \in Z^+$ if $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ where $a \neq 1$.

(f) $A^n = \begin{bmatrix} a^n & n a^{n-1} \\ 0 & a^n \end{bmatrix}$ for every positive integer n, if $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$.

(g) $A^n = \begin{bmatrix} 1 & n & \dfrac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$, $n \in N$ if it is given that $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(h) $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all the positive integers n, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Q10. **(a)** If $A = \text{diag}(a \quad b \quad c)$, show that $A^n = \text{diag}(a^n \quad b^n \quad c^n)$ for all positive integers n.

(b) If A and B are square matrices of the same order such that AB = BA, then prove by using induction that $AB^n = B^n A$. Further, prove that $(AB)^n = A^n B^n$ for all $n \in N$.

EXERCISE 1 J

BASED ON APPLICATIONS

Q01. In XII class examination, 25 students from school A and 35 students from school B appeared. Only 20 students from each school could get through the examination. Out of them, 15 students from school A and 10 students from school B secured full marks. Write down this information in matrix from.

Q02. Let matrix $A = \begin{bmatrix} 8 & 16 \\ 32 & 48 \end{bmatrix}$, where first row represents the number of table fans and second row represents the number of ceiling fans which two manufacturing units x and y makes in one day. Compute 7A and, state what does it represents?

Q03. Two farmers Ramkrishna and Hari Prasad cultivated three varieties of rice namely Basmati, Permal and Naura.
The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices ‘A’ and ‘B’ :

$$\begin{array}{cc} & \text{September Sales (in Rupees)} \\ & \begin{array}{ccc}\text{Basmati} & \text{Permal} & \text{Naura}\end{array} \\ A = & \begin{pmatrix} 10000 & 20000 & 30000 \\ 50000 & 30000 & 10000 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{Ramkrishna} \\ \text{Hari Prasad} \end{array}$$

and,

$$\begin{array}{cc} & \text{October Sales (in Rupees)} \\ & \begin{array}{ccc}\text{Basmati} & \text{Permal} & \text{Naura}\end{array} \\ B = & \begin{pmatrix} 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{Ramkrishna} \\ \text{Hari Prasad} \end{array}$$

(i) Find the combined sale in September and October for each farmer in each variety.
(ii) Find the decrease in sales from September to October.
(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.
(iv) Which farmer gets more profit in the overall sales for both the months?
(v) Which farmer in your opinion is more resourceful and why?

Q04. In Chennai, there are 50 colleges. Each has 30 teachers, 20 non-teaching staffs, 1 Principal, 2 Vice Principals and, 5 peons. Express this information in the form of a column matrix.
Using scalar multiplication, find the total number of posts of each type in the colleges.